Engineering Notes

Compressibility Effects on Maximum Range Cruise at Constant Altitude

Damián Rivas* and Alfonso Valenzuela†
Universidad de Sevilla, 41092 Seville, Spain

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Introduction

IN A recent paper, Pargett and Ardema [1] analyze the problem of range maximization in cruise at constant altitude as a *singular optimal control* problem. The same problem is also analyzed using different approaches by Miele [2] and Torenbeek [3], who consider the case of quasi-steady flight. In [1], the singular arc that defines the optimal path is studied; in this study, a simple aircraft model defined by a parabolic drag polar of constant coefficients is considered and only one value of cruise altitude.

Many other analyses of cruise optimization can be found in the literature, with different flight constraints and performance indices; see, for instance, the study of Menon [4], in which both speed and altitude are allowed to vary, and the many references therein.

In this work, we generalize the analysis of [1] considering a general drag polar, so that compressibility effects can be taken into account. The singular arc in this general case is obtained and is particularized for both asymmetric and symmetric compressible parabolic drag polars. The results show that compressibility effects are very important; the differences with the incompressible case are shown to be not only quantitative, but also qualitative. We also analyze the influence of cruise altitude on the optimal paths, showing that it can be important from a qualitative point of view. The maximum range is calculated as a function of cruise altitude; in particular, the best altitude that leads to the largest maximum range is obtained. Results are presented for a model of a Boeing 767-300ER.

Problem Formulation

The optimal control problem is formulated in [1]. However, it is summarized here for completeness. It is desired to maximize the range for a given fuel load or, equivalently, to minimize the following performance index,

$$J = -\int_0^{t_f} V \mathrm{d}t \tag{1}$$

with final time t_f unspecified, subject to the following constraints:

$$\dot{V} = \frac{1}{m}(T - D) \qquad \dot{m} = -cT \tag{2}$$

which are the equations of motion for cruise at constant altitude and constant heading. In these equations, the drag is a general known function $D(V, m)$, which takes into account the remaining equation of motion $L = mg$. The thrust $T(V)$ is given by $T = \pi T_M(V)$, where π models the throttle setting, $\pi_m \leq \pi \leq 1$, and $T_M(V)$ is a known function. The specific fuel consumption $c(V)$ is also a known function. Thus, this problem has two states (speed V and mass m) and one control (π). The initial (m_0) and final (m_f) values of the aircraft mass are given.

This problem is similar to that of maximizing altitude for a sounding rocket for a fixed amount of propellant, analyzed by Bryson and Ho [5].

The Hamiltonian of this problem is given by

$$H = -V + \lambda_V \frac{1}{m}(\pi T_M - D) - \lambda_m c \pi T_M \tag{3}$$

where λ_V and λ_m are the adjoints. Because H is linear in the control variable, one can write $H = F + S\pi$, with

$$F = -V - \lambda_V \frac{D}{m} \qquad S = \left(\lambda_V \frac{1}{m} - \lambda_m c\right) T_M \tag{4}$$

The function S is called the *switching* function. The necessary conditions for optimality are stated in [5].

Singular Arc

Because H is not an explicit function of time, one has $H =$ constant on the optimal trajectory. Moreover, because t_f is not specified, one has the additional condition $H(t_f) = 0$. Therefore, one has $H(t) = 0$ on the optimal path. The singular control is obtained when the switching function is zero ($S = 0$) on an interval of time; hence, because $H = 0$, one also has $F = 0$. On that interval of time, one has $\dot{S} = 0$ as well. The singular arc is defined by the three equations: $F = 0$, $S = 0$, and $\dot{S} = 0$ (see [5]).

Taking into account the state equations (2) and the following adjoint equations [5],

$$\begin{aligned} \dot{\lambda}_V &= -\frac{\partial H}{\partial V} = 1 - \lambda_V \frac{1}{m}\left(\pi \frac{dT_M}{dV} - \frac{\partial D}{\partial V}\right) \\ &\quad + \lambda_m \pi \left(\frac{dc}{dV} T_M + c \frac{dT_M}{dV}\right) \\ \dot{\lambda}_m &= -\frac{\partial H}{\partial m} = \lambda_V \left[\frac{1}{m^2}(\pi T_M - D) + \frac{1}{m}\frac{\partial D}{\partial m}\right] \end{aligned} \tag{5}$$

the function $\dot{S}$ is given by

$$\begin{aligned} \dot{S} &= -\frac{dT_M}{dV}\left(\frac{\lambda_V}{m} - \lambda_m c\right)\frac{D}{m} \\ &\quad + \frac{T_M}{m}\left[1 + \lambda_V\left(\frac{1}{m}\frac{\partial D}{\partial V} - c\frac{\partial D}{\partial m} + c\frac{D}{m}\right) + \lambda_m \frac{dc}{dV} D\right] \end{aligned} \tag{6}$$

Note that the terms in the control variable π have cancelled out of this equation.

Hence, the three equations that define the singular arc ($F = 0$, $S = 0$, and $\dot{S} = 0$) reduce to

$$\begin{gathered} V + \lambda_V \frac{D}{m} = 0 \qquad \frac{\lambda_V}{m} - \lambda_m c = 0 \\ 1 + \lambda_V\left(\frac{1}{m}\frac{\partial D}{\partial V} - c\frac{\partial D}{\partial m} + c\frac{D}{m}\right) + \lambda_m \frac{dc}{dV} D = 0 \end{gathered} \tag{7}$$

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*Professor, Escuela Superior de Ingenieros, Department of Aerospace Engineering; drivas@us.es.

†Assistant Professor, Escuela Superior de Ingenieros, Department of Aerospace Engineering.

The singular arc is obtained after eliminating the adjoints λ_V and λ_m from these equations. One obtains the following expression, which generalizes that obtained in [1]:

$$D\left(1 - Vc - \frac{V}{c}\frac{dc}{dV}\right) - V\frac{\partial D}{\partial V} + Vcm\frac{\partial D}{\partial m} = 0 \tag{8}$$

For a symmetric parabolic drag polar of constant coefficients, this expression reduces to the one obtained in [1]. Equation (8) defines a singular line in the state space [i.e., the (V, m) space], which is, in fact, the locus of possible points in the state space on which optimal paths can lie.

Dimensionless Singular Arc

Let us consider a general drag polar $C_D = C_D(C_L, M)$, where C_D and C_L are the drag and lift coefficients, and $M = V/a$ is the Mach number (a being the speed of sound). From the definition of C_D and C_L, one has $D = \frac{1}{2}\rho V^2 S_W C_D(C_L, M)$ and $C_L = (mg/\frac{1}{2}\rho V^2 S_W)$, where the equation of motion $L = mg$ has been taken into account (ρ is the density and S_W is the wing surface area). The following partial derivatives are obtained:

$$\frac{\partial D}{\partial V} = \frac{1}{2}\rho V S_W\left(2C_D - 2C_L\frac{\partial C_D}{\partial C_L} + M\frac{\partial C_D}{\partial M}\right) \qquad \frac{\partial D}{\partial m} = g\frac{\partial C_D}{\partial C_L} \tag{9}$$

Also, let us consider a specific fuel-consumption coefficient $C = C(M)$, defined by $c = (a/L_H)C$, where L_H is the fuel latent heat. The following derivative is obtained:

$$\frac{dc}{dV} = \frac{1}{L_H}\frac{dC}{dM} \tag{10}$$

In general, C is a function of M and the thrust coefficient $C_T = T/(W_{\mathrm{TO}}\delta)$, where W_{TO} is the reference takeoff weight and $\delta = p/p_{\mathrm{SL}}$ is the pressure ratio (p_{SL} being the reference sea-level pressure). However, the dependence of C with C_T is, in practice, very weak and can be neglected (see [2]).

Substituting the previous derivatives into Eq. (8), one gets

$$C_D\left(1 + \frac{a^2}{L_H}MC + \frac{M}{C}\frac{dC}{dM}\right) - \left(2 + \frac{a^2}{L_H}MC\right)C_L\frac{\partial C_D}{\partial C_L} + M\frac{\partial C_D}{\partial M} = 0 \tag{11}$$

which is the general dimensionless expression for the singular arc.

As indicated in [1], in general, one has $Vc \ll 1$: that is, $(a^2/L_H)MC \ll 1$; hence, Eq. (11) can be simplified into

$$C_D\left(1 + \frac{M}{C}\frac{dC}{dM}\right) - 2C_L\frac{\partial C_D}{\partial C_L} + M\frac{\partial C_D}{\partial M} = 0 \tag{12}$$

This equation is also obtained by Torenbeek [3] in the case of quasi-steady flight, by a different approach, for the case in which the overall engine efficiency is independent of the corrected thrust T/δ, and an expression equivalent to this one is obtained by Miele [2] in the case of quasi-steady flight, by a different approach, for the particular case of a symmetric parabolic drag polar.

Optimal Singular Control

The function $\ddot{S}$ depends linearly on the control variable [say, $\ddot{S} = A(V, m) + B(V, m)\pi$]; therefore, because one also has $\ddot{S} = 0$ (where $S = 0$), the singular control is obtained from $A(V, m) + B(V, m)\pi = 0$. After laborious manipulations, one gets the following:

$$\pi = \frac{D}{T_M}(1 + VcG(V, m)) \tag{13}$$

where $G(V, m)$ is a function of order unity (given by a lengthy expression not included for the sake of brevity).

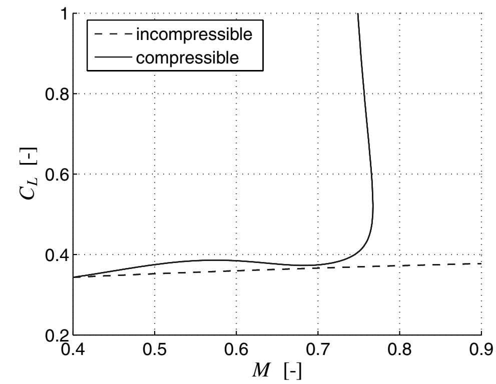

Fig. 1 Singular arc in the C_L–M plane.

In the case of $Vc \ll 1$, one has $\pi = D/T_M$; that is, one has the well-known cruise hypothesis $T = D$ (this is also shown in [1] for the particular case of a polar of constant coefficients and specific fuel consumption independent of V).

An additional condition [the generalized Legendre–Clebsch condition (see [6])] establishes that for the singular control to be optimal, one must have $-(\partial\ddot{S}/\partial\pi) \geq 0$; that is, one must have the following necessary condition:

$$B(V, m) \leq 0 \tag{14}$$

which has been shown to be satisfied numerically, for the aircraft model defined in the following section, for the entire singular arc represented in Fig. 1.

Results

The aircraft model considered in this paper for the numerical applications is that of a Boeing 767-300ER (a typical twin-engine, wide-body, transport aircraft), with wing surface area of 283.3 m^2, maximum takeoff weight (MTOW) of 186,880 kg, and maximum fuel weight of 73,635 kg [7]. The drag polar defined by Cavcar and Cavcar [8] is considered; it is given by

$$C_D = \left(C_{D_{0,i}} + \sum_{j=1}^{5} k_{0j}K^j(M)\right) + \left(C_{D_{1,i}} + \sum_{j=1}^{5} k_{1j}K^j(M)\right)C_L + \left(C_{D_{2,i}} + \sum_{j=1}^{5} k_{2j}K^j(M)\right)C_L^2 \tag{15}$$

where

$$K(M) = \frac{(M - 0.4)^2}{\sqrt{1 - M^2}} \tag{16}$$

The incompressible drag-polar coefficients for the model aircraft are $C_{D_{0,i}} = 0.01322$, $C_{D_{1,i}} = -0.00610$, and $C_{D_{2,i}} = 0.06000$, and the compressible coefficients are given in Table 1. This polar is valid for $M \geq 0.4$; for $M \leq 0.4$, the incompressible drag polar applies [obtained by setting $K = 0$ in Eq. (15)].

For the specific fuel-consumption coefficient, the linear model defined by Mattingly et al. [9] (and approximately depicted by Miele [2]) is considered; it is given by

$$C = c_{\mathrm{SL}}\frac{L_H}{a_{\mathrm{SL}}}(1.0 + 1.2M) \tag{17}$$

where c_{SL} is the specific fuel consumption at sea level and $M = 0$. For the CF6-80C2 engines of the model aircraft, we take a representative value of $c_{\mathrm{SL}} = 9.0 \times 10^{-6}$ kg/(s · N).[‡] For the fuel latent heat, we take $L_H = 43 \times 10^6$ J/kg.

[‡]Data available online at http://www.geae.com/engines/commercial/cf6/cf6-80c2.html [retrieved June 2009].

Table 1 Compressible drag-polar coefficients for the model aircraft

j	1	2	3	4	5
k_{0j}	0.0067	−0.1861	2.2420	−6.4350	6.3428
k_{1j}	0.0962	−0.7602	−1.2870	3.7925	−2.7672
k_{2j}	−0.1317	1.3427	−1.2839	5.0164	0.0000

With respect to the thrust, the following simple analytical model is considered for the thrust coefficient (it is based on Mattingly et al.'s [9] model for the dependence on M and takes into account the increase of C_T with altitude [10]):

$$C_T = \frac{T_{\mathrm{SL}}}{W_{\mathrm{TO}}}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}(1-0.49\sqrt{M})\frac{1}{\theta} \tag{18}$$

where $\gamma = 1.4$ (ratio of specific heats) and T_{SL} is the maximum thrust at sea level and for $M = 0$ for the cruise rating. This expression defines the maximum thrust $T_M = W_{\mathrm{TO}}\delta C_T$. For the two CF6-80C2 engines of the model aircraft, the following representative value is considered: $T_{\mathrm{SL}} = 5.0 \times 10^5$ N (see footnote ‡).

With respect to the atmosphere, the ISA model is considered.

Singular Arc

The singular arc in the C_L–M plane defined by Eq. (11) is plotted in Fig. 1. In this figure, three curves for different values of altitude are represented, although the differences cannot be noticed, which corroborates the fact that the altitude-dependent term $(a^2/L_H)MC$ is indeed negligible. If one considers the quasi-steady case analyzed by Miele [2] and Torenbeek [3], defined by Eq. (12), the singular arc would also show no difference with those represented in Fig. 1. $C_L(M)$ is a multivalued function in the interval $0.7111 < M < 0.7673$. This function is composed of two branches, which branch off at $(C_L, M) = (0.7673, 0.5220)$. The figure shows that there is a maximum value of the Mach number that can be obtained: namely, $M = 0.7673$ (for our aircraft model). The larger values of C_L in Fig. 1 are very high for cruise; however, for the optimal paths actually flown by the aircraft (represented at the end of this section), C_L ranges from 0.3762 (at 9000 m) to 0.6499 (at 11,000 m), which are appropriate for the cruise regime.

From the definition of C_L, one has

$$q_0\delta_A M^2 C_L(M) = W \tag{19}$$

where W is the aircraft weight, $q_0 = \frac{1}{2}\gamma p_{\mathrm{SL}} S_W$, and δ_A is the pressure ratio at the given altitude (say, h_A). This expression defines the singular arc in the W–M plane for each value of h_A. Letting $\omega = W/(q_0\delta_A)$, the singular arc in the ω–M plane is represented in Fig. 2, which is, in practice, independent of the altitude. Hence, the maximum value of M is the same for all altitudes.

It must be noted that in the case of a symmetric parabolic drag polar studied by Miele [2], the maximum value of the Mach number is not obtained, but M increases monotonously, with ω tending asymptotically to a limit value that is below 0.8. If we consider a symmetric drag polar, obtained by neglecting the term proportional to C_L in Eq. (15), a result analogous to that reported by Miele is obtained, as shown in Fig. 3. Hence, the type of drag-polar function considered affects the results.

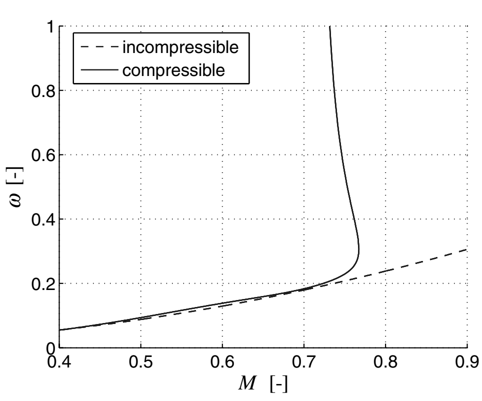

Fig. 2 Singular arc in the ω–M plane.

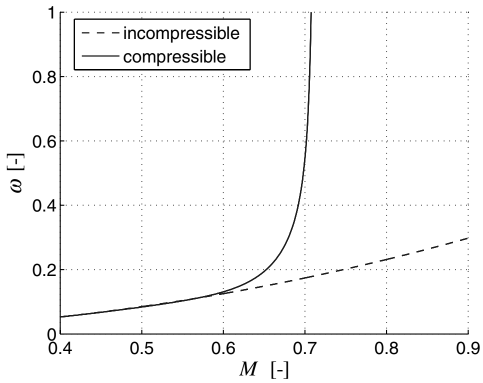

Fig. 3 Singular arc in the ω–M plane for a symmetric polar.

As one can see in Fig. 2, for low values of ω, until the turning point is reached, the singular arc defines an increase of M with W, whereas for large values of ω, after the turning point, it defines a decrease. Note that for given aircraft weight, low values of ω correspond to low altitudes, and vice versa. Therefore, for given initial and final values of W, at low altitudes one has the well-known behavior of M decreasing as fuel is consumed, whereas at high altitudes one can have the opposite. This behavior is shown in Fig. 4 for different values of h_A, where the optimal paths are represented by thick lines superposed on the singular arcs. In this simulation, we have taken $W_i = 1700$ kN and $W_f = 1150$ kN, which is a fuel load for a cruise of 550 kN (approximately 30% of MTOW). Thus, we can conclude

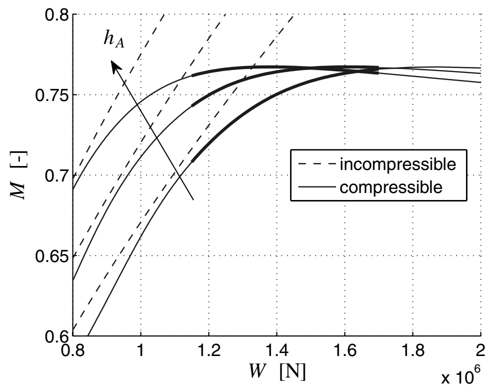

Fig. 4 Optimal paths (h_A = 9000, 10000, and 11,000 m).

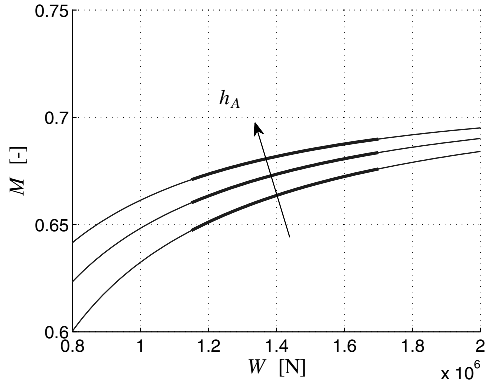

Fig. 5 Optimal paths for a symmetric polar (h_A = 9000, 10000, and 11,000 m).

that the cruise altitude has a qualitative influence on the results. In the case of a symmetric drag polar, the results show the decrease of Mach number as fuel is consumed, at any altitude, as shown in Fig. 5.

Implicit in the results presented in Figs. 4 and 5 is that the entire cruise can follow the singular arc. The inequality constraint $\pi_m \leq \pi \leq 1$ is satisfied for those optimal paths. The optimal singular control for the optimal paths represented in Fig. 4 is depicted in Fig. 6. Note that the required thrust decreases as fuel is consumed.

Compressibility Effects

To analyze compressibility effects, we consider the incompressible drag polar obtained by making $K = 0$ in Eq. (15). The incompressible singular arcs have been plotted in Figs. 1–4; one obtains the result that the Mach number increases strongly with aircraft weight (this is also the result obtained in [1] for an incompressible model of a Boeing 747-400). Comparing with the compressible results, we can see that the incompressible drag polar overestimates the optimal Mach number, which can become quite large (even supersonic), whereas, as shown before, a compressible drag polar defines a maximum value of M (or, in the case of a symmetric drag polar, a limit value of M) in the subsonic region. Thus, compressibility effects prevent the Mach number from increasing unrealistically. We conclude that between compressible and incompressible results there is no agreement, neither quantitative nor qualitative.

Maximum Range

The range is given by [using the second state equation (3)]

$$R = \int_0^{t_f} V \mathrm{d}t = -\frac{1}{g}\int_{W_i}^{W_f} \frac{V}{cT} \mathrm{d}W = \frac{1}{g}\int_{W_f}^{W_i} \frac{a(h_A)M}{c(M, h_A)T(M, h_A, W)} \mathrm{d}W \tag{20}$$

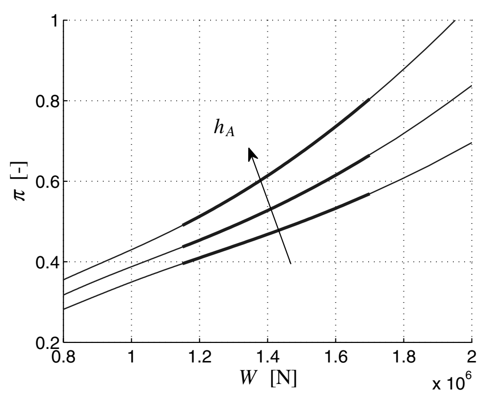

Fig. 6 Optimal singular control ($h_A = 9000$, 10000, and 11,000 m).

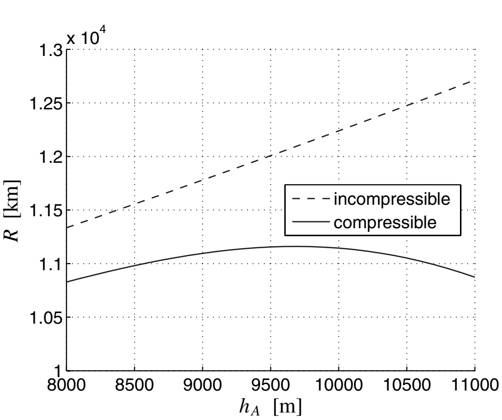

Fig. 7 Maximum range.

where the thrust $T = \pi T_M$ is defined by the singular control (13). Once the optimal Mach number $M(W, h_A)$ is obtained from the singular arc (19), the integral can be calculated, yielding the maximum range $R_{\max}$, which is plotted in Fig. 7 as a function of cruise altitude h_A. The results show that there is a best altitude at which the maximum range is largest: namely, $(h_A)_{\text{best}} = 9683$ m, with $(R_{\max})_{\text{best}} = 11,160$ km.

The result for an incompressible drag polar is also shown in Fig. 7, which we can see is in complete disagreement with the compressible case, overestimating the value of $R_{\max}$, especially at large altitudes.

In [1], a comparison with a standard cruise at constant altitude and constant speed is made: for a given value of altitude, by considering an operational value of cruise speed (larger than the optimal-path speeds), an increase in range for the optimal path of 9.8% is reported. This is a very large value caused by the fact that the chosen cruise speed is much larger than the speed for best range in a standard cruise (a fact also indicated by the authors). Our results show that if at each altitude the speed for best range is chosen, the optimal-path maximum range is just slightly larger than the best standard cruise range: larger by less than 1% in all cases.

A much smaller flight time for the standard cruise is also reported in [1], as compared with that of the optimal path. Again, this is due to the particularity of the chosen cruise speed. If at each altitude the speed for best range is chosen for the standard cruise, which is close to the optimal-path speeds, the flight times in both flight regimes are quite close to one another.

Conclusions

Maximum range cruise at constant altitude has been analyzed as a singular optimal control problem for an aircraft model with a general compressible drag polar. The singular arc has been obtained in this general case. It has been shown that compressibility effects must be taken into account to properly describe the behavior of modern high-speed subsonic transport aircraft. Comparison with an incompressible drag polar has shown large differences, both quantitative and qualitative.

The singular arc presents a maximum subsonic value of Mach number, practically the same for all altitudes (or a limit subsonic value in the case of a symmetric parabolic drag polar), whereas in the incompressible case the Mach number increases without limit with aircraft weight, becoming even supersonic. The maximum range for the optimal path as a function of altitude presents a maximum, which defines the best altitude at which the maximum range is largest, whereas in the incompressible case the maximum range increases monotonously with altitude, leading to unrealistically large values of range.

The influence of altitude on the optimal paths has been shown to be important as well, caused by the maximum Mach number defined by the singular arc. For a given aircraft, the optimal path for cruise at low altitudes requires that the Mach number decrease as fuel is consumed, whereas at high altitudes one has the opposite behavior; at intermediate altitudes one may have a combination of both. However, in the case of a *symmetric* parabolic drag polar, in which the singular arc presents an asymptotic limit value for the Mach number, but not a maximum value, one has the same behavior irrespective of the cruise altitude: namely, a decrease of Mach number as the aircraft weight decreases.

It must be emphasized that in this work, a constrained regime has been considered: namely, flight at constant altitude (of interest from the air traffic control point of view). The optimization problem has then defined a constrained maximum (it is well known that improved performance is obtained flying, for instance, a *cruise climb*, for which altitude slightly increases).

Other optimality criteria, such as minimizing direct operating cost (that is, a combination of fuel consumption and flight time, of interest for airlines) or minimizing fuel consumption with a final-time constraint (for instance, a given time at the top-of-descent point, of interest for arrival managers) can be also analyzed. These analyses are left for future work.

Acknowledgments

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